Indian Statistical Institute
M.Math I Year Second Semester Back Paper Exam 2003-2004
Algebra II

Time: 3 hours Date:21-07-04 Max. Marks : 100

Note: Answer all questions.

1. State the Eisenstein's criteria for the irreducibility of a polynomial in $Q[x]$.
   Determine if $f(x) = x^2 + 27x + 213$ is irreducible over the following fields:
   (i) Q (ii) $\mathbb{F}_5$ [10]

2. Define a Noetherian ring and a Noethreian module. Give examples.
   Prove that if N is a submodule of an R-module M, then M is Noetherian if and only if N and M/N are Noetherian. [10]

3. Define localisation of a ring. Let R be a commutative ring. Prove that an ideal I of R is a prime ideal if and only if it is maximal with respect to the property of not meeting some multiplicatively closed subset of R. [10]

4. Define the characteristic of a field. Give an example of an infinite field of characteristic $p > 0$. If k is a field of characteristic $p > 0$ and $f[x] \in k[x]$ is irreducible, then show that $f(x)$ divides $X^{q^n} - X$ if and only if the degree of f divides n. [10]

5. Define splitting field of a polynomial in $\mathbb{F}$[X] where $\mathbb{F}$ is a field. Prove that for any $f \in \mathbb{F}$[X], splitting field of f exists and is unique up to an isomorphism of fields. [10]

6. Let $a \in K = \mathbf{Q}(e^{2i\pi/n})$. If α is a root of $X^n - a$, prove that $K(\alpha)$ is a cyclic extension of K, of some degree d dividing n. [10]

7. State (do not prove) precisely what the natural numbers n are for which the regular n-gon can be constructed by a ruler and a compass. Use this to deduce that an angle of 60 degrees can be constructed but an angle of 20 degrees cannot be constructed by ruler and compass. [10]

8. Let K be an extension of a field F. Let $x_1, \ldots, x_n \in K$ such that $\{x_1, \ldots, x_n\}$ is algebraically independent over F. Given any sequence $r_1, \ldots, r_n$ of positive integers, show that $\{x_1^{r_1}, \ldots, x_n^{r_n}\}$ is also algebraically independent over F. [10]

9. (a) Let $n > 0$ be an integer. Find the integral closure of the ring of integers $\mathbb{Z}$ in the ring $Q[\sqrt{n}] = \{a + b\sqrt{n} | a, b \in Q\}$.
(b) Let $R \subset T$ be domains with R integrally closed in T, and let $S \subset R$ be a multiplicatively closed subset of R. Prove that R_S is integrally closed in T_S. [10]

10. State the Galois correspondence theorem, including the definitions of the relevant concentps. Find the Galois group of the splitting field of the polynomial $X^3 + X^2 - 3$ over Q. [10]

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